Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The number n of positive integers less than p and prime to it is

$$n=p(1-1/a)(1-1/b)(1-1/c)(1-1/d).....$$

The sum of the squares of all such numbers is

$$S_2=\tfrac{1}{3}p^3(1-1/a)(1-1/b)(1-1/c)(1-1/d).....+\tfrac{1}{6}p(1-a)(1-b)(1-c)(1-d).....$$

and the sum of the cubes is

$$S_3=\tfrac{1}{4}p^4(1-1/a)(1-1/b)(1-1/c)(1-1/d).....+\tfrac{1}{4}p^2(1-a)(1-b)(1-c)(1-d).....$$

$$\therefore A_2=S_2/n=\tfrac{1}{3}p^2+\tfrac{1}{6}\cdot\frac{(1-a)(1-b)(1-c)(1-d).....}{(1-1/a)(1-1/b)(1-1/c)(1-1/d).....}$$

$$=\tfrac{1}{3}p^2+\tfrac{1}{6}B, \text{ suppose.}$$

$$A_3=S_3/n=\tfrac{1}{4}p^3+\tfrac{1}{4}pB.$$

$$\therefore B=6A_2-2p^2=\frac{4A_3-p^3}{p}.$$

$$\therefore 6A_2p-2p^3=4A_3-p^3, \text{ or } 6A_2p-4A_3=p^3, \text{ or } p^3-6A_2p+4A_3=0.$$

179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left(\frac{q^2}{p}-\frac{p^3}{5q}\right)=0.$$

No solution of this problem has been received.

180. Proposed by the late JOSIAH H. DRUMMOND.

If r/s is such a value of p as makes $m/(p^2-2)$ integral, prove that $(3r+4s)/(2r+3s)$ is another such value, so that an indefinite number of integral values may be obtained.

Also, if r/s is such a value of p as makes $2m/(p^2-2)$ integral, prove that $2(r+s)/(r+2s)$ is also such a value.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

When $p=r/s$, $m/(p^2-2)=ms^2/(r^2-2s^2)$.....(1).

When $p=(3r+4s)/(2r+3s)$, $m/(p^2-2)=m(2r+3s)^2/(r^2-2s^2)$.....(2).

Since (1) is integral, (2) is also, for we can take $m=n(r^2+2s^2)$.

When $p=r/s$, $2m/(p^2-2)=2ms^2/(r^2-2s^2)$.....(3).

When $p=2(r+s)/(r+2s)$, $2m/(p^2-2)=m(r+2s)^2(r^2-2s^2)$.....(4).

Since (3) is integral, (4) is also.

Also solved by the *PROPOSER.*

GEOMETRY.

200. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of eight points of contact of the four common tangents of two concentric coaxial ellipses.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/a^2+y^2/b^2=1$, and $x^2/c^2+y^2/d^2=1$, be the equations to the ellipses, $c>a$, $b>d$.

Since $xx_1/a^2+yy_1/b^2=1$, and $xx_2/c^2+yy_2/d^2=1$, are two equations for the same line, we get $x_2=c^2x_1/a^2$, $y_2=d^2y_1/b^2$.

$\therefore x_1^2/a^2+y_1^2/b^2=1$, and $c^2x_1^2/a^4+d^2y_1^2/b^4=1$, give

$$x_1=\frac{a^2\surd[b^2-d^2]}{\surd[b^2c^2-a^2d^2]}=a^2m \text{ (suppose)},\ y_1=\frac{b^2\surd[c^2-a^2]}{\surd[b^2c^2-a^2d^2]}=b^2n,$$

$$x_2=c^2m,\ y_2=d^2n.$$

Hence $mx\pm ny\pm1=0$ represents the four common tangents. While $(c^2m,\ d^2n)$; $(a^2m,\ b^2n)$; $(-a^2m,\ b^2n)$; $(-c^2m,\ d^2n)$; $(-c^2m,\ -d^2n)$; $(a^2m,\ -b^2n)$; $(c^2m,\ -d^2n)$ are the eight points of contact. These eight points are situated on an ellipse, which is the locus required.

Let $x^2/A^2+y^2/B^2=1$ be this ellipse. Then $c^2m/A^2+d^4n^2/B^2=1$, also $a^4m^2/A^2+b^4n^2/B^2=1$.

$$\therefore A^2=\frac{(b^4c^4-a^4d^4)m^2}{b^4-d^4}=\frac{b^2c^2+a^2d^2}{b^2+d^2},\ B^2=\frac{(b^4c^4-a^4d^4)n^2}{c^4-a^4}=\frac{b^2c^2+a^2d^2}{a^2+c^2}.$$

$$\therefore (b^2+d^2)x^2+(a^2+c^2)y^2=b^2c^2+a^2d^2 \text{ is the locus.}$$

201. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

Two plane sections of a right circular cone have their major axes AA', aa' coplanar, and Aa on one generator equal to $A'a'$ on the other. The projections of the sections on any plane perpendicular to the axis are confocal.